

3-SUBMERSIONS FROM QR-HYPERSURFACES OF QUATERNIONIC KÄHLER MANIFOLDS

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ABSTRACT. In this paper we study 3-submersions from a QR-hypersurface of a quaternionic Kähler manifold onto an almost quaternionic hermitian manifold. We also prove the non-existence of quaternionic submersions between quaternionic Kähler manifolds which are not locally hyper-Kähler.

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1. INTRODUCTION

In [22] B. Watson introduced the notion of 3-submersion, as a Riemannian submersion from an almost contact metric manifold onto an almost quaternionic manifold, which commutes with the structure tensors of type (1,1). In [10] and [11], this concept has been extended in quaternionic setting. In this paper we study 3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds, we give an example and obtain some obstructions to the existence of quaternionic submersions.

The study of QR-submanifolds of a quaternionic Kähler manifold was initiated by A. Bejancu [4]. Among all submanifolds of a quaternionic Kähler manifold, QR-submanifolds have been intensively studied by several authors [1, 3, 5, 9, 13, 14, 17, 19, 20]. In Section 2 one recalls the definitions and basic properties of quaternionic manifolds and QR-submanifolds of a quaternionic Kähler manifold.

On another hand, R. Güneş, B. Şahin and S. Keleş [9] have shown that a QR-submanifold admits an almost contact 3-structure in some conditions. In Section 3 we see that on an orientable hypersurface of a quaternionic Kähler manifold there exists a natural almost contact metric 3-structure. This result will allow us to define the concept of QR 3-submersion. In Section 4 we obtain some properties for this kind of submersions and give an example. In the last section we prove the non-existence of quaternionic submersions between quaternionic Kähler non-locally hyper-Kähler manifolds.

2. PRELIMINARIES

Let M be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying:

$$(1) \quad \begin{cases} J_\alpha^2 = -Id, \forall \alpha = \overline{1, 3} \\ J_1 J_2 = -J_2 J_1 = J_3 \end{cases}$$

Then the bundle σ is called an almost quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ . Moreover, (M, g) is said to be an almost

quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $n = 4m$.

A Riemannian metric g is said to be adapted to the quaternionic structure σ if it satisfies:

$$(2) \quad g(J_\alpha X, J_\alpha Y) = g(X, Y), \forall \alpha = \overline{1, 3}$$

for all vector fields X, Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ . Moreover, (M, σ, g) is said to be an almost quaternionic hermitian manifold.

If the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g , then (M, σ, g) is said to be a quaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that:

$$(3) \quad \begin{cases} \nabla_X J_1 = \omega_3(X)J_2 - \omega_2(X)J_3 \\ \nabla_X J_2 = -\omega_3(X)J_1 + \omega_1(X)J_3 \\ \nabla_X J_3 = \omega_2(X)J_1 - \omega_1(X)J_2 \end{cases}$$

for any vector field X on M . In particular, if $\omega_1 = \omega_2 = \omega_3 = 0$, then (M, σ, g) is said to be a locally hyper-Kähler manifold.

We remark that any quaternionic Kähler manifold M is an Einstein manifold, provided that $\dim M > 4$. Moreover, M is irreducible (if $\text{Ric} \neq 0$) or locally hyper-Kähler manifold (if $\text{Ric} = 0$) (see [2, 6, 12, 21]).

Let (M, σ, g) be an almost quaternionic hermitian manifold. If $X \in T_p M, p \in M$, then the 4-plane $Q(X)$ spanned by $\{X, J_1 X, J_2 X, J_3 X\}$ is called a quaternionic 4-plane. A 2-plane in $T_p M$ spanned by $\{X, Y\}$ is called half-quaternionic if $Q(X) = Q(Y)$.

The sectional curvature for a half-quaternionic 2-plane is called quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c . It is well-known that a quaternionic Kähler manifold (M, σ, g) is a quaternionic space form (denoted $M(c)$) if and only if its curvature tensor is:

$$(4) \quad \begin{aligned} R(X, Y)Z &= \frac{c}{4} \{g(Z, Y)X - g(X, Z)Y + \sum_{\alpha=1}^3 [g(Z, J_\alpha Y)J_\alpha X \\ &\quad - g(Z, J_\alpha X)J_\alpha Y + 2g(X, J_\alpha Y)J_\alpha Z]\} \end{aligned}$$

for all vector fields X, Y, Z on M and any local basis $\{J_1, J_2, J_3\}$ of σ .

Let $(\overline{M}, \sigma, g)$ be a quaternionic Kähler manifold and let M be a real submanifold of \overline{M} . Then M is called QR-submanifold if there exists a vector subbundle D of the normal bundle TM^\perp such that we have:

- i. $J_\alpha(D_p) = D_p, \forall p \in M, \forall \alpha = \overline{1, 3}$;
- ii. $J_\alpha(D_p^\perp) \subset T_p M, \forall p \in M, \forall \alpha = \overline{1, 3}$, where D^\perp is the complementary orthogonal bundle to D in TM^\perp (see [4]).

3. QR-HYPERSURFACES AND ALMOST CONTACT METRIC 3-STRUCTURES

Let M be an orientable hypersurface of a quaternionic Kähler manifold \overline{M} and ξ a unit normal field on M . If we take $D = 0$, then $D^\perp = TM^\perp$ and we conclude that M is a QR-submanifold of \overline{M} .

Let $\{J_\alpha\}_{\alpha=\overline{1,3}}$ and $\{J'_\alpha\}_{\alpha=\overline{1,3}}$ two local bases defined on coordinate neighborhoods \overline{U} and \overline{U}' , with $\overline{U} \cap \overline{U}' \neq \emptyset$. Then, on \overline{U} :

$$\xi_\alpha = -J_\alpha \xi, \forall \alpha = \overline{1,3},$$

defines tangent vector fields to M and similarly, on \overline{U}' :

$$\xi'_\alpha = -J'_\alpha \xi, \forall \alpha = \overline{1,3},$$

defines tangent vector fields to M .

Moreover, on $\overline{U} \cap \overline{U}'$ we have:

$$\xi'_\alpha = \sum_{\beta=1}^3 c_{\alpha\beta} \xi_\beta, \forall \alpha = \overline{1,3},$$

where $C = (c_{\alpha\beta})_{\alpha,\beta=\overline{1,3}} \in SO(3)$. Thus, we obtain a distribution \mathcal{V} on M , which is locally generated by $\{\xi_\alpha\}_{\alpha=\overline{1,3}}$. Let \mathcal{H} be the orthogonal complementary distribution to \mathcal{V} with respect to the Riemannian metric g induced by \overline{g} on M . We remark that for each $p \in M$, \mathcal{H}_p is J_α -invariant, $\forall \alpha = \overline{1,3}$.

We recall that the distribution \mathcal{V} is integrable if and only if M is a mixed geodesic QR-hypersurface of \overline{M} , i.e:

$$(5) \quad B(U, X) = 0, \forall U \in \Gamma(\mathcal{V}), \forall X \in \Gamma(\mathcal{H}),$$

where B is the second fundamental form of M in \overline{M} (see [4]).

Definition 3.1. [7] Let M be a differentiable manifold equipped with a triple (ϕ, ξ, η) , where ϕ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M . If we have:

$$(6) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1$$

then we say that (ϕ, ξ, η) is an almost contact structure on M .

Definition 3.2. [16] Let M be a differentiable manifold which admits three almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha), \forall \alpha = \overline{1,3}$ satisfying the following conditions:

$$(7) \quad \eta_\alpha(\xi_\beta) = 0, \forall \alpha \neq \beta,$$

$$(8) \quad \phi_\alpha(\xi_\beta) = -\phi_\beta(\xi_\alpha) = \xi_\gamma,$$

$$(9) \quad \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha = \eta_\gamma$$

and

$$(10) \quad \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta = \phi_\gamma,$$

where in (8), (9) and (10), (α, β, γ) is an even permutation of $(1, 2, 3)$. Then the manifold M is said to have an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$.

Definition 3.3. [16] Let (M, g) be a Riemannian manifold, endowed with an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ such that:

$$(11) \quad \eta_\alpha(X) = g(X, \xi_\alpha), \forall \alpha = \overline{1,3}$$

and

$$(12) \quad g(\phi_\alpha X, \phi_\alpha Y) = g(X, Y) - \eta_\alpha(X) \eta_\alpha(Y), \forall \alpha = \overline{1,3}$$

for all vector fields X, Y on M . Then we say that M admits an almost contact metric 3-structure.

Definition 3.4. [7] An almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ on a Riemannian manifold (M, g) is said to be a 3-cosymplectic structure if:

$$(13) \quad (\nabla_X \phi_\alpha)(Y) = 0, (\nabla_X \eta_\alpha)(Y) = 0, \forall \alpha = \overline{1,3}.$$

Let M be an orientable hypersurface of a quaternionic Kähler manifold \overline{M} . If $S : TM \rightarrow \mathcal{H}$ is the canonical projection, then, we have that any local vector field X on M is expressed as follows:

$$(14) \quad X = SX + \sum_{\alpha=1}^3 \eta_\alpha(X) \xi_\alpha,$$

where:

$$(15) \quad \eta_\alpha(X) = g(X, \xi_\alpha), \forall \alpha = \overline{1,3}.$$

From (14) we have:

$$(16) \quad J_\alpha X = J_\alpha SX + \sum_{\beta=1}^3 \eta_\beta(X) J_\alpha \xi_\beta, \forall \alpha = \overline{1,3}.$$

From (16) we obtain the decomposition:

$$(17) \quad J_\alpha X = \phi_\alpha X + F_\alpha X,$$

where $\phi_\alpha X$ is the tangential part of $J_\alpha X$, given by:

$$(18) \quad \phi_\alpha X = J_\alpha SX + \eta_\beta(X) \xi_\gamma - \eta_\gamma(X) \xi_\beta,$$

and $F_\alpha X$ is the normal part of $J_\alpha X$, given by:

$$(19) \quad F_\alpha X = \eta_\alpha(X) \xi,$$

for all $\alpha = \overline{1,3}$, where (α, β, γ) is an even permutation of $(1, 2, 3)$.

By straightforward computations, we can easily see that $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_\alpha$, defined by (15), (18) and (19), is an almost contact metric 3-structure on M (see also [9]) and so we have the next result.

Proposition 3.5. *Any QR-hypersurface of a quaternionic Kähler manifold admits a natural almost contact metric 3-structure.*

4. 3-SUBMERSIONS OF QR-HYPERSURFACES

Definition 4.1. Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold \overline{M} , endowed with the natural almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$, given by Proposition 3.5 and let (M', σ', g') be an almost quaternionic hermitian manifold. We say that a Riemannian submersion $\pi : M \rightarrow M'$ is a QR 3-submersion if the following conditions are satisfied:

- i. $\text{Ker} \pi_* = \mathcal{V}$;
- ii. For each $p \in M$, $\sigma'_{\pi(p)}$ admits a canonical local basis $\{J'_1, J'_2, J'_3\}$ such that:

$$\pi_* \phi_\alpha = J'_\alpha \pi_*, \forall \alpha = \overline{1,3}.$$

Remark 4.2. We recall that the sections of \mathcal{V} , respectively \mathcal{H} , are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \rightarrow M'$ determines two (1,2) tensor field T and A on M , by the formulas:

$$(20) \quad T(E, F) = T_E F = h \nabla_{v_E} v F + v \nabla_{v_E} h F$$

and respectively:

$$(21) \quad A(E, F) = A_E F = v \nabla_{hE} hF + h \nabla_{hE} vF$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projection (see [15, 18]).

We remark that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2}v[X, Y]$ reflecting the complete integrability of the horizontal distribution \mathcal{H} .

An horizontal vector field X on M is said to be basic if X is π -related to a vector field X' on M' . It is clear that every vector field X' on M' has a unique horizontal lift X to M and X is basic.

Remark 4.3. If $\pi : M \rightarrow M'$ is a Riemannian submersion and X, Y are basic vector fields on M , π -related to X' and Y' on M' , then we have the next properties (see [6, 8, 18]):

- i. $h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;
- ii. $h(\nabla_X Y)$ is a basic vector field π -related to $\nabla'_{X'} Y'$, where ∇ and ∇' are the Levi-Civita connections on M and M' ;
- iii. $[E, U] \in \Gamma(\mathcal{V})$, $\forall U \in \Gamma(\mathcal{V})$ and $\forall E \in \Gamma(TM)$.

Proposition 4.4. *Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $(\overline{M}, \overline{\sigma}, \overline{g})$ and let (M', σ', g') be an almost quaternionic hermitian manifold. If $\pi : M \rightarrow M'$ is a QR 3-submersion, then the distributions \mathcal{V} and \mathcal{H} are invariant by ϕ_α , $\forall \alpha = \overline{1, 3}$.*

Proof. Let $V \in \Gamma(\mathcal{V})$. Then, we have:

$$\pi_* \phi_\alpha V = J'_\alpha \pi_* V = 0,$$

and so we conclude that $\phi_\alpha(\mathcal{V}) \subset \mathcal{V}$.

On the other hand, for any $X \in \Gamma(\mathcal{H})$, we have:

$$g(\phi_\alpha X, V) = g(J_\alpha X, V) = -g(X, J_\alpha V) = 0,$$

and thus we obtain $\phi_\alpha(\mathcal{H}) \subset \mathcal{H}$. \square

Theorem 4.5. *Let $\pi : M \rightarrow M'$ be a QR 3-submersion such that the canonical almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ on M is a 3-cosymplectic structure. Then M' is locally hyper-Kähler.*

Proof. For any local basic vector fields X, Y on M , π -related with X' and Y' on M' , from (13) we have:

$$(22) \quad \nabla_X \phi_\alpha Y - \phi_\alpha \nabla_X Y = 0, \forall \alpha = \overline{1, 3}.$$

and from (22) we deduce:

$$(23) \quad \pi_*(\nabla_X \phi_\alpha Y) - \pi_* \phi_\alpha \nabla_X Y = 0, \forall \alpha = \overline{1, 3}.$$

Then, since Y is a basic vector field π -related with Y' , also $\phi_\alpha Y$ is basic and π -related with $J'_\alpha Y'$ and taking account of Definition 4.1 and Remark 4.3, we obtain from (23) that we have:

$$\nabla'_{X'} J'_\alpha Y' - J'_\alpha \nabla'_{X'} Y' = 0, \forall \alpha = \overline{1, 3},$$

and thus we conclude that $(\nabla'_{X'} J'_\alpha) Y' = 0$, and so M' is locally hyper-Kähler. \square

Corollary 4.6. *Let M be a totally geodesic QR-hypersurface of a quaternionic Kähler manifold $(\overline{M}, \overline{\sigma}, \overline{g})$ and (M', σ', g') be an almost quaternionic hermitian manifold. If $\pi : M \rightarrow M'$ is a QR 3-submersion such that ξ_1, ξ_2 and ξ_3 are parallel in M , then M' is locally hyper-Kähler.*

Proof. In this case $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ is a 3-cosymplectic structure on M (see [9]) and the proof is obvious from Theorem 4.5. \square

Theorem 4.7. *Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $(\overline{M}, \overline{\sigma}, \overline{g})$, (M', σ', g') be an almost quaternionic hermitian manifold and $\pi : M \rightarrow M'$ be a QR 3-submersion. If the natural almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ on M is 3-cosymplectic, then the fibre submanifolds are totally geodesic immersed and the horizontal distribution is integrable.*

Proof. Since M is 3-cosymplectic we have:

$$(24) \quad \nabla_U \phi_\alpha V = \phi_\alpha \nabla_U V, \forall \alpha = \overline{1,3},$$

for $\forall U, V \in \Gamma(\mathcal{V})$. Taking the horizontal components, we obtain:

$$(25) \quad T_U \phi_\alpha V = \phi_\alpha T_U V, \forall \alpha = \overline{1,3}$$

which immediately implies:

$$(26) \quad T_U V = -T_{\phi_\alpha U} \phi_\alpha V, \forall \alpha = \overline{1,3}.$$

From (26), taking account of (1), we obtain $T = 0$. Similarly we obtain $A = 0$ and the proof is now complete, via Remark 4.2. \square

Let M be an orientable submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$. We say that M is a totally umbilical submanifold of \overline{M} if the second fundamental form h of M satisfied:

$$(27) \quad h(E, F) = g(E, F)H$$

$\forall E, F \in \Gamma(TM)$, where H is the mean curvature vector field on M . Moreover, if H is nonzero and parallel in the normal bundle TM^\perp , then M is called an extrinsic sphere.

By using the Gauss equation, (4) and the Gray-O'Neill equation (see [6, 8, 17, 18]), we can easily prove the next result.

Theorem 4.8. *Let M be a QR extrinsic hypersphere of a flat quaternionic Kähler manifold $(\overline{M}, \overline{\sigma}, \overline{g})$ and let (M', σ', g') be another quaternionic Kähler manifold. If $\pi : M \rightarrow M'$ is a QR 3-submersion, then M' is a quaternionic space form.*

Example. Let S^{4m+3} be the standard hypersphere in R^{4m+4} . Then the canonical mapping $\pi : S^{4m+3} \rightarrow P^m(H)$ is a QR 3-submersion.

5. QUATERNIONIC SUBMERSIONS

Definition 5.1. [10] Let (M, σ, g) and (N, σ', g') be two almost quaternionic hermitian manifolds. A map $f : M \rightarrow N$ is said to be a (σ, σ') -holomorphic map at a point $x \in M$ if for any $J \in \sigma_x$ exists $J' \in \sigma'_{f(x)}$ such that $f_* \circ J = J' \circ f_*$. Moreover, we say that f is a (σ, σ') -holomorphic map if f is a (σ, σ') -holomorphic map at each point $x \in M$.

Definition 5.2. [11] Let (M, σ, g) and (N, σ', g') be two almost quaternionic hermitian manifolds. A Riemannian submersion $\pi : M \rightarrow N$ which is a (σ, σ') -holomorphic map is called a quaternionic submersion.

Theorem 5.3. *Let $\pi : (M, \sigma, g) \rightarrow (N, \sigma', g')$ be a quaternionic submersion such that (M, σ, g) is a quaternionic Kähler manifold. Then (N, σ', g') is a quaternionic Kähler manifold.*

Proof. If we consider $X_*, Y_* \in \Gamma(TN)$ such that $\pi_*X = X_*, \pi_*Y = Y_*$, where $X, Y \in \Gamma(TM)$, we obtain:

$$\begin{aligned}
 (\nabla'_{X_*} J'_\alpha)Y_* &= \nabla'_{X_*}(J'_\alpha Y_*) - J'_\alpha(\nabla'_{X_*} Y_*) \\
 &= \nabla'_{\pi_*X}(J'_\alpha \pi_*Y) - J'_\alpha(\nabla'_{\pi_*X} \pi_*Y) \\
 &= \nabla'_{\pi_*X}(\pi_*(J_\alpha Y)) - J'_\alpha \pi_*(h \nabla_X Y) \\
 &= \pi_*(h \nabla_X (J_\alpha Y)) - \pi_*(J_\alpha (h \nabla_X Y)) \\
 (28) \qquad &= \pi_*((\nabla_X J_\alpha)Y).
 \end{aligned}$$

Since (M, σ, g) is a quaternionic Kähler manifold we have (3) and we can define 1-forms $\omega'_1, \omega'_2, \omega'_3$ on N by:

$$(29) \qquad \omega'_\alpha(X_*) \circ \pi = \omega_\alpha(X), \forall \alpha \in \{1, 2, 3\},$$

for any local vector field X_* on N and X a real basic vector field on M such that $\pi_*X = X_*$.

From (3), (28) and (29) we deduce for all $\alpha \in \{1, 2, 3\}$:

$$(30) \qquad (\nabla'_{X_*} J'_\alpha)Y_* = \omega'_{\alpha+2}(X_*)J'_{\alpha+1}Y_* - \omega'_{\alpha+1}(X_*)J'_{\alpha+2}Y_*,$$

for any local vector fields X_*, Y_* on N , where the indices are taken from $\{1, 2, 3\}$ modulo 3. Thus we conclude that (N, σ', g') is a quaternionic Kähler manifold. \square

Corollary 5.4. *Let $\pi : (M, \sigma, g) \rightarrow (N, \sigma', g')$ be a quaternionic submersion such that (M, σ, g) is a quaternionic Kähler manifold. Then, both (M, σ, g) and (N, σ', g') are locally hyper-Kähler manifolds.*

Proof. In this case we have that the vertical and horizontal distributions are both integrable (see [11]) and so we can easily conclude that (M, σ, g) is a locally hyper-Kähler manifold. The assertion follows now from the above Theorem. \square

Corollary 5.5. *There are no quaternionic submersions between quaternionic Kähler manifold which are not locally hyper-Kähler.*

Proof. The assertion is obvious from the above Corollary. \square

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